Maximum rank webs are not necessarily almost Grassmannizable

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Abstract *We present an example of a* 6*-web* $W(6,3,2)$ *of codimension two and of maximum rank on a six-dimensional manifold which is not almost Grassmannizable.*

0. Introduction

In 1984, during the Problem Session in the meeting on web geometry at the Mathematisches Forschungsinstitut Oberwolfach, Goldberg posed the following problem (see [8]):

6. *Every* d*-web* $W(d,n,r)$ *of maximum* r*-rank is almost Grassmannizable. Is it true or wrong?"*

Little in his paper [7] related the Chern and Griffiths approach in studying the rank problem and Grassmannization and algebraization problems for webs based on the presence of abelian equations on the web with the Akivis and Goldberg approach based on the notion of the almost Grassmannizable web. He wrote in the introduction: "The major purpose of this paper is to relate these two approaches by showing that, in rough terms, if d is large enough (relative to r,n) so that $\pi(d,n,r)>1$, then every maximum rank web $W(d,n,r)$ is almost-Grassmannizable. This answers (part of) a question posed by Goldberg at the 1984 Oberwolfach Conference on web geometry."

Little proved in [7] that *if* $r>d(n-1)+2$*, then every maximum* r*-rank web* $W(d,n,r)$ *is almost Grassmannizable.*

In [7] Little also considered maximum r-rank webs $W(d,n,r)$ with $d=r(n-1)+2$. In particular, he proved that "the maximum 2-rank webs $W(6,3,2)$ are also almost Grassmannizable" (see his Example 1). It appears that this last Little's result was wrong.

When Goldberg presented for publication the first version of his paper [5], where he considered the maximum rank webs $W(6,3,2)$, he applied the above Little's result and deduced that these kind of webs are always almost Grassmannizable.

A referee suspected that Little's result is incorrect and provided a counterexample. Goldberg recognized that the counterexample is correct and communicated this counterexample to Little. Little double-checked his proof and found that the proof was incorrect. He discovered that the error in his proof is in Corollary 3.6, and this happened because of rather tricky a general position question. As a result, his Example 1, in which he claimed that the webs $W(6,3,2)$ of maximum rank are almost Grassmannizable, was incorrect. The referee's counterexample was a counterexample to Little's Corollary 3.6.

However, Little discussed the case of webs $W(6,3,2)$ as a special case separately from the main results which requires that the number of abelian equations be two or more. The main result of his paper and his Example 2 giving an almost Grassmannizable web $W(r+2,2,r)$ of maximum rank one were correct.

In the revised version of the paper [5], Goldberg changed the reference to Little's paper [7] by the additional assumption that *the maximum rank webs $W(d,n,r)$ in question are almost Grassmannizable.*

While looking recently at the paper [5], the author of the present paper recognized the importance of the referee counterexample. This counterexample looks easy but it is not trivial and was not known so far.

The purpose of this article is to present and study in detail the above mentioned referee counterexample of a maximum 2-rank web $W(6,3,2)$ which is not almost Grassmannizable.

1. Codimension Two Webs on a 6-Dimensional Differentiable Manifold

1. In an open domain D of a differentiable manifold X^6 of dimension six a 6-*web* $W(6,3,2)$ *of codimension two* is given by six codimension two foliations X_ξ, $\xi = 1,2,3,4,5,6$, if the tangent 4-planes to the leaves $V_\xi \subset X_\xi$ through a point in D are in general position.

Two webs $W(6,3,2)$ and $\widetilde{W}(6,3,2)$ are *equivalent* to each other if there exists a local diffeomorphism $\phi : D \to \widetilde{D}$ of their domains transferring the foliations of W into the foliations of $\widetilde{W}$.

Let X_ξ, $\xi = 1,\dots,6$, be six foliations of parallel 4-planes in an affine space $\mathbb{A}^6$ of dimension six. Suppose that the 4-planes of different foliations are in general position. Such a 6-web is called *parallel*. A web $W(6,3,2)$ which is equivalent to a parallel web $W(6,3,2)$ is called *parallelizable*.

The foliations X_ξ, $\xi = 1,\dots,6$, of the web $W(6,3,2)$ can be given by six completely integrable systems of Pfaffian equations

$$\underset{\xi}{\omega}^i = 0, \quad \xi = 1,\dots,6; \quad i = 1,2, \tag{1}$$

where the forms $\underset{\alpha}{\omega}^i$, $\alpha = 1,2,3$, are the basis forms of the manifold X^6 and

$$-\underset{4}{\omega}^i = \underset{1}{\omega}^i + \underset{2}{\omega}^i + \underset{3}{\omega}^i, \tag{2}$$

$$-\underset{a}{\omega}^i = \underset{a1}{\lambda}{}^i_j \underset{1}{\omega}^j + \underset{a2}{\lambda}{}^i_j \underset{2}{\omega}^j + \underset{3}{\omega}^i, \quad a = 5,6 \tag{3}$$

(see [4] for webs $W(d,2,r)$), where the quantities $\underset{a\hat{\alpha}}{\lambda}{}^i_j$, $i,j = 1,2$, form an $(1,1)$-tensor for any $a = 5,6$ and $\hat{\alpha} = 1,2$, and these four tensors $\underset{a\hat{\alpha}}{\lambda}{}^i_j$ are distinct and satisfy some additional conditions. They are called the *basis affinors* of a web $W(6,3,2)$ (cf. [4]).

2. We define now the notion of the almost Grassmann structure. To this end, we need to define first the notion of Segre cones (see [1], Section 4). Consider the Plücker mapping of the Grassmannian $G(2,4)$ of 2-planes of a projective space $\mathbb{P}^4$ onto an algebraic manifold $\Omega(2,4)$ of dimension six of a projective space $\mathbb{P}^9$. This mapping can be constructed by means of the Grassmann coordinates of a 2-plane

L in P^4 which are the determinants of order three of the matrix

$$\begin{pmatrix} x_1^1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \end{pmatrix}$$

composed of the coordinates of the basis points x_1, x_2, x_3 of the 2-plane L. The Grassmann coordinates are connected by a set of quadratic relations that define the manifold $\Omega(2,4)$ in the space P^9 (see [6], Chap. 7, §6). We will say that this manifold carries the *Grassmann structure* and denote this manifold shortly by Ω.

Let L_1 and L_2 be two 2-planes in $\mathbb{P}^4$ meeting in the straight line K. They generate a linear pencil S of 2-planes $\lambda L_1 + \mu L_2$. A rectilinear generator of the manifold Ω corresponds to this pencil. All the 2-planes of the pencil S belong to a 3-plane M. This pencil, and consequently the corresponding straight line in Ω, is completely determined by a pair K and M, $K \subset M$.

Consider a bundle of 2-planes, i.e., a set of all 2-planes passing through a fixed straight line K. On the manifold Ω, to this bundle there corresponds a two-dimensional plane generator ξ^2. On the other hand, on Ω, to a family of 2-planes belonging to a fixed 3-plane M, there corresponds a three-dimensional plane generator η^3. Thus, the manifold Ω carries two families of plane generators of dimensions two and three, respectively.

If the straight line K and the plane M are incident, $K \subset M$, the plane generators ξ^2 and η^3 defined by these planes, meet along a straight line. If they are not incident, then the generators ξ^2 and η^3 have no common points.

Let us consider a fixed 2-plane L in $\mathbb{P}^4$. It contains a two-parameter family of straight lines K. Therefore, the two-parameter family of generators ξ^2 passes through the point $p \in \Omega$ corresponding to L. On the other hand, a one-parameter family of 3-planes M passes through the same plane L. Consequently, a one-parameter family of generators η^3 passes through the point p. Furthermore, any two generators ξ^2 and η^3 passing through p meet along a straight line. It follows that all the plane generators ξ^2 and η^3 passing through the point p, form a cone whose projectivization is the *Segre manifold* $S(1,2)$ in the projective space $\mathbb{P}^5$. The Segre manifold carries two families of plane generators of dimensions one and two and can be considered as the projective embedding of the Cartesian product of two projective spaces $\mathbb{P}^1$ and $\mathbb{P}^2$ into the space $\mathbb{P}^5$:

$$S(1,2) : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5.$$

The above described cone, whose projectivization is the Segre manifold $S(1,2)$, is called the *Segre cone* and is denoted by $C_p(2,3)$. This cone is the intersection of the manifold Ω and its tangent space $T_p(\Omega)$, whose dimension is the same as that of Ω, namely six. In the space $\mathbb{P}^5$, the set of all two-dimensional planes intersecting a fixed 2-plane L along straight lines corresponds to the cone $C_p(2,3)$.

Therefore, with each point p of the algebraic manifold $\Omega \subset P^9$, there is connected the Segre cone $C_p(2,3)$ with vertex p located in the tangent space $T_p(\Omega)$, and the generators of this cone are generators of the manifold Ω.

An *almost Grassmann structure* on a manifold X^6 is a smooth field of Segre cones (or their projectivizations) in the tangent spaces TX^6.

Consider 4-subwebs [4, 1, 2, 3], [5, 1, 2, 3], and [6, 1, 2, 3] defined by the first three foliations X_1, X_2, X_3 of a web $W(6,3,2)$ and the foliations X_4, X_5, and

X_6, respectively. Each of these 4-subwebs defines an almost Grassmann structure. If all three almost Grassmann structure so defined coincide, a web $W(6,3,2)$ is said to be an *almost Grassmann web*. Webs $W(6,3,2)$ that are equivalent to an almost Grassmann web are called *almost Grassmannizable*. We denote such webs by $AGW(6,3,2)$. At each point all two-fold intersections of the tangent spaces to the web leaves are 2-dimensional generators of the Segre cone at that point.

For almost Grassmannizable webs $AGW(6,3,2)$, we have

$$\underset{a\hat{\alpha}}{\lambda}{}^i_j = \underset{a\hat{\alpha}}{\lambda}\,\delta^i_j, \quad a = 5,6; \quad \hat{\alpha} = 1,2 \tag{4}$$

(cf. [4]). Using equations (4), we can write equations (3) in the form

$$-\underset{a}{\omega}{}^i = \underset{a1}{\lambda}\,\underset{1}{\omega}{}^i + \underset{a2}{\lambda}\,\underset{2}{\omega}{}^i + \underset{3}{\omega}{}^i, \quad a = 5,6. \tag{5}$$

The coefficients $\underset{a1}{\lambda}$ and $\underset{a2}{\lambda}$ in (4) satisfy certain inequalities which are implied by the fact that the system of equations (1), (2), and (5) must be solvable with respect to any six forms $\underset{\xi}{\omega}{}^i$, $\underset{\eta}{\omega}{}^j$, $\underset{\zeta}{\omega}{}^k$, $\xi,\eta,\zeta = 1,\dots,6$; $\xi \neq \eta,\zeta$; $\eta \neq \zeta$ (see [5]).

The forms $\underset{\alpha}{\omega}{}^i$, $\alpha = 1,2,3$; $i = 1,2$, satisfy the following structure equations:

$$d\underset{\alpha}{\omega}{}^i - \underset{\alpha}{\omega}{}^j \wedge \omega^i_j = \sum_{\beta\neq\alpha} \underset{\alpha\beta}{a}{}^i_{jk}\,\underset{\alpha}{\omega}{}^j \wedge \underset{\beta}{\omega}{}^k \tag{6}$$

(see [4]), where the quantities $\underset{\alpha\beta}{a}{}^i_{jk}$ form the *torsion tensor* of the web $AGW(6,3,2)$.

On the manifold X^6, a web $W(6,3,2)$ defines an affine connection γ which is determined by the forms $\omega^\xi = \{\underset{\alpha}{\omega}{}^i\}$, $\xi,\eta = 1,2,3,4,5,6$; $\alpha = 1,2,3$; $i = 1,2$, and

$$\begin{pmatrix} (\omega^i_j) & O_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & (\omega^i_j) & O_{2\times 2} \\ O_{2\times 2} & O_{2\times 2} & (\omega^i_j) \end{pmatrix}$$

(see [4]), where $(\omega^j_i) = \begin{pmatrix} \omega^1_1 & \omega^2_1 \\ \omega^1_2 & \omega^2_2 \end{pmatrix}$ and $O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

3. Suppose that the leaves of the ξth foliation of a web $W(6,3,2)$ are given as level sets of functions $u^i_\xi(x)$:

$$u^i_\xi(x) = \text{const.}, \qquad \xi = 1,\dots,6.$$

The functions $u^i_\xi(x)$ are defined up to a local diffeomorphism in the space of $u^i_\xi(x)$.

An exterior 2-equation of the form

$$\sum_{\xi=1}^{6} f_\xi(u^j_\xi)\,du^1_\xi \wedge du^2_\xi = 0, \qquad j = 1,2,$$

is said to be an *abelian* 2-*equation*. The maximum number R_2 of linearly independent abelian 2-equations admitted by a web $W(6,3,2)$ is called the 2-*rank* of the web $W(6,3,2)$.

It follows from the definition that the coefficients f_ξ are constant on the leaves of the ξth foliation of $W(6,3,2)$.

If there exists an upper bound $\pi_2(6,3,2)$ of R_2, then $R_2 \leq \pi_2(6,3,2)$. Chern and Griffiths [2] found the general formula for $\pi(d,n,r)$. It follows from their formula that $\pi_2(6,3,2) = 1$.

It was proved in [5] that the condition

$$\lambda_{51}\lambda_{62}(1 - \lambda_{52} - \lambda_{61}) = \lambda_{52}\lambda_{61}(1 - \lambda_{51} - \lambda_{62}) \tag{7}$$

is a necessary condition for a web $AGW(6,3,2)$ to be of maximum rank one.

2. Referee's Counterexample

4. Suppose that $(x^1,\, x^2,\, x^3,\, y_4,\, y_5,\, y_6)$ are coordinates in $\mathbb{R}^6$ and A is any non-singular 3×3 matrix of real numbers. Define the linear functions $x^4,\, x^5,\, x^6,\, y_1,\, y_2,\, y_3$ on $\mathbb{R}^6$ as follows:

$$(x^4, x^5, x^6) = (x^1, x^2, x^3)\, A \ \text{ and } \ (y_1, y_2, y_3) = -(y_4, y_5, y_6)\, A^T.$$

If

$$A = \begin{pmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{pmatrix},$$

then

$$x^{3+\alpha} = a_\beta^\alpha\, x^\beta, \quad \alpha, \beta = 1,2,3, \tag{8}$$

and

$$y_\alpha = -a_\alpha^\beta\, y_{3+\beta}, \quad \alpha, \beta = 1,2,3. \tag{9}$$

Equation (9) can be written as

$$y_{3+\alpha} = -b_\alpha^\beta\, y_\beta, \tag{10}$$

where $B = (b_\alpha^\beta)$ is the inverse matrix of the matrix A.

Differentiation of (8) and (9) gives

$$dx^{3+\alpha} = a_\beta^\alpha\, dx^\beta, \tag{11}$$

and

$$dy_\alpha = -a_\alpha^\beta\, dy_{3+\beta}. \tag{12}$$

Using (11) and (12), we can easily find that

$$dx^\xi \wedge dy_\xi = 0, \quad \xi = 1,2,3,4,5,6. \tag{13}$$

Hence equations

$$dx^\xi = 0, \quad dy_\xi = 0, \quad \xi = 1,2,3,4,5,6,$$

define a set of foliations X_ξ of codimension two on $\mathbb{R}^6$, i.e., a 6-web $W(6,3,2)$ admitting an abelian 2-equation (13).

First, we prove that the 6-web we have constructed is parallelizable.

Proposition 1. *The 6-web defined by equations* (8) *and* (10) *is parallelizable.*

Proof. In fact, each of the foliations X_ξ of the constructed 6-web are defined by equations

$$dx^\xi = 0, \ \ dy_\xi = 0, \ \ \xi = 1, 2, 3, 4, 5, 6,$$

where the index ξ is fixed, or

$$x^\xi = \text{const.}, \ \ y_\xi = \text{const.}$$

Each of these two equations represents a foliation of parallel hyperplanes in $\mathbb{R}^6$, and the leaves of X_ξ are parallel 4-planes that are intersections of parallel hyperplanes of the above two foliations of hyperplanes. $\square$

5. According to [2], a 6-web can admit at most one abelian 2-equation. For a generic matrix A, the web normals $dx^1 \wedge dy_1, dx^2 \wedge dy_2, dx^3 \wedge dy_3, dx^4 \wedge dy_4, dx^5 \wedge dy_5$, and $dx^6 \wedge dy_6$ are, in some sense, in the most general position which allows there to be an abelian 2-equation (13). So, we constructed a 6-web $W(6, 3, 2)$ of maximum 2-rank. Note that because the matrix A is generic, its entries (which are constants) are not connected by any relation, i.e., there is a 9-parameter family of such matrices.

We prove now that the 6-web constructed above is not almost Grassmannizable.

Theorem 2. *The* 6*-web defined by equations* (8) *and* (10) *is not almost Grassmannizable.*

Proof. To prove the theorem, first we reduce equations (11) and (12) to the form (2) and (3).

Denote

$$\underset{\alpha}{\omega}^1 = dx^\alpha, \ \ \underset{\alpha}{\omega}^2 = dy_\alpha = -a^\beta_\alpha \, dy_{3+\beta}. \tag{14}$$

Solving equations (12), we find that

$$-dy_{3+\alpha} = \sum_\beta b^\beta_\alpha \, dy_\beta, \tag{15}$$

where $B = (b^\beta_\alpha)$ is the inverse matrix of the matrix A.

Denote

$$\underset{4}{\omega}^1 = dx^4, \ \ \underset{4}{\omega}^2 = dy_4. \tag{16}$$

Define

$$-\underset{\alpha}{\overline{\omega}}^1 = a^1_\alpha \, \underset{\alpha}{\omega}^1, \ \ \underset{\alpha}{\overline{\omega}}^2 = b^\alpha_1 \, \underset{\alpha}{\omega}^2 \ \ \text{(no summation over } \alpha). \tag{17}$$

If we suppress the bar over $\underset{\alpha}{\omega}^i$, then equations (16) take the form (2):

$$-\underset{4}{\omega}^i = \underset{1}{\omega}^i + \underset{2}{\omega}^i + \underset{3}{\omega}^i. \tag{18}$$

It follows from (11), (15), and (17) that

$$\left\{\begin{array}{ll} -dx^5 = \dfrac{a^2_1}{a^1_1} \, \underset{1}{\omega}^1 + \dfrac{a^2_2}{a^1_2} \, \underset{2}{\omega}^1 + \dfrac{a^2_3}{a^1_3} \, \underset{3}{\omega}^1, & -dx^6 = \dfrac{a^3_1}{a^1_1} \, \underset{1}{\omega}^1 + \dfrac{a^3_2}{a^1_2} \, \underset{2}{\omega}^1 + \dfrac{a^3_3}{a^1_3} \, \underset{3}{\omega}^1, \\ -dy_5 = \dfrac{b^1_2}{b^1_1} \, \underset{1}{\omega}^2 + \dfrac{b^2_2}{b^2_1} \, \underset{2}{\omega}^2 + \dfrac{b^3_2}{b^3_1} \, \underset{3}{\omega}^2, & -dy_6 = \dfrac{b^1_3}{b^1_1} \, \underset{1}{\omega}^2 + \dfrac{b^2_3}{b^2_1} \, \underset{2}{\omega}^2 + \dfrac{b^3_3}{b^3_1} \, \underset{3}{\omega}^2. \end{array}\right. \tag{19}$$

Equations (19) can be written as

$$
(20)\qquad \begin{cases}
-\dfrac{a_3^1}{a_3^2}\,dx^5 = \dfrac{a_1^2a_3^1}{a_1^1a_3^2}\,\underset{1}{\omega}{}^1 + \dfrac{a_2^2a_3^1}{a_2^1a_3^2}\,\underset{2}{\omega}{}^1 + \underset{3}{\omega}{}^1, \\[2mm]
-\dfrac{a_3^1}{a_3^3}\,dx^6 = \dfrac{a_1^3a_3^1}{a_1^1a_3^3}\,\underset{1}{\omega}{}^1 + \dfrac{a_2^3a_3^1}{a_2^1a_3^3}\,\underset{2}{\omega}{}^1 + \underset{3}{\omega}{}^1, \\[2mm]
-\dfrac{b_1^3}{b_2^3}\,dy_5 = \dfrac{b_2^1b_1^3}{b_1^1b_2^3}\,\underset{1}{\omega}{}^2 + \dfrac{b_2^2b_1^3}{b_1^2b_2^3}\,\underset{2}{\omega}{}^2 + \underset{3}{\omega}{}^2, \\[2mm]
-\dfrac{b_1^3}{b_3^3}\,dy_6 = \dfrac{b_3^1b_1^3}{b_1^1b_3^3}\,\underset{1}{\omega}{}^2 + \dfrac{b_3^2b_1^3}{b_1^2b_3^3}\,\underset{2}{\omega}{}^2 + \underset{3}{\omega}{}^2.
\end{cases}
$$

Define

$$
(21)\qquad \underset{5}{\omega}{}^1 = \frac{a_3^1}{a_3^2}\,dx^5,\ \ \underset{5}{\omega}{}^2 = \frac{b_1^3}{b_2^3}\,dy_5,\ \ \underset{6}{\omega}{}^1 = \frac{a_3^1}{a_3^3}\,dx^6,\ \ \underset{6}{\omega}{}^2 = \frac{b_1^3}{b_3^3}\,dy_6.
$$

Then equations (20) take the form

$$
(22)\qquad \begin{cases}
-\underset{5}{\omega}{}^1 = \dfrac{a_1^2a_3^1}{a_1^1a_3^2}\,\underset{1}{\omega}{}^1 + \dfrac{a_2^2a_3^1}{a_2^1a_3^2}\,\underset{2}{\omega}{}^1 + \underset{3}{\omega}{}^1, \quad -\underset{6}{\omega}{}^1 = \dfrac{a_1^3a_3^1}{a_1^1a_3^3}\,\underset{1}{\omega}{}^1 + \dfrac{a_2^3a_3^1}{a_2^1a_3^3}\,\underset{2}{\omega}{}^1 + \underset{3}{\omega}{}^1, \\[2mm]
-\underset{5}{\omega}{}^2 = \dfrac{b_2^1b_1^3}{b_1^1b_2^3}\,\underset{1}{\omega}{}^2 + \dfrac{b_2^2b_1^3}{b_1^2b_2^3}\,\underset{2}{\omega}{}^2 + \underset{3}{\omega}{}^2, \quad -\underset{6}{\omega}{}^2 = \dfrac{b_3^1b_1^3}{b_1^1b_3^3}\,\underset{1}{\omega}{}^2 + \dfrac{b_3^2b_1^3}{b_1^2b_3^3}\,\underset{2}{\omega}{}^2 + \underset{3}{\omega}{}^2.
\end{cases}
$$

Comparing equations (22) with equations (3), we find that

$$
(23)\qquad \begin{cases}
\underset{51}{\lambda}{}^1_i = \dfrac{a_1^2a_3^1}{a_1^1a_3^2}\,\delta_i^1, \quad \underset{52}{\lambda}{}^1_i = \dfrac{a_2^2a_3^1}{a_1^2a_3^2}\,\delta_i^1, \quad \underset{51}{\lambda}{}^2_i = \dfrac{b_2^1b_1^3}{b_1^1b_2^3}\,\delta_i^2, \quad \underset{52}{\lambda}{}^2_i = \dfrac{b_2^2b_1^3}{b_1^1b_2^3}\,\delta_i^2, \\[2mm]
\underset{61}{\lambda}{}^1_i = \dfrac{a_1^3a_3^1}{a_1^1a_3^3}\,\delta_i^1, \quad \underset{62}{\lambda}{}^1_i = \dfrac{a_2^3a_3^1}{a_1^2a_3^3}\,\delta_i^1, \quad \underset{61}{\lambda}{}^2_i = \dfrac{b_3^1b_1^3}{b_1^1b_3^3}\,\delta_i^2, \quad \underset{62}{\lambda}{}^2_i = \dfrac{b_3^2b_1^3}{b_2^1b_3^3}\,\delta_i^2.
\end{cases}
$$

These equations show that *a web* $W(6,3,2)$ *of maximum* 2*-rank defined by equations* (8) *and* (10) *has constant basis affinors.*

Next suppose that our 6-web under consideration is almost Grassmannizable. This will be the case if and only if equations (22) have the form (5) or the functions $\underset{a\hat{\alpha}}{\lambda}{}^i_j$ have the form (4). Comparing (4) and (23), we find that our web is almost Grassmannizable if and only if the following four conditions hold:

$$
(24)\qquad \begin{cases}
\underset{51}{\lambda} = \dfrac{a_1^2a_3^1}{a_1^1a_3^2} = \dfrac{b_2^1b_1^3}{b_1^1b_2^3}, \quad \underset{52}{\lambda} = \dfrac{a_2^2a_3^1}{a_1^2a_3^2} = \dfrac{b_2^2b_1^3}{b_1^1b_2^3}, \\[2mm]
\underset{61}{\lambda} = \dfrac{a_1^3a_3^1}{a_1^1a_3^3} = \dfrac{b_3^1b_1^3}{b_1^1b_3^3}, \quad \underset{62}{\lambda} = \dfrac{a_2^3a_3^1}{a_1^2a_3^3} = \dfrac{b_3^2b_1^3}{b_2^1b_3^3}.
\end{cases}
$$

A straightforward calculation shows that these four conditions are equivalent to the condition which can be written in three equivalent forms:

$$\begin{vmatrix} a_1^1 & a_1^2 & a_1^1 a_1^2 a_2^3 a_3^3 \\ a_2^1 & a_2^2 & a_2^1 a_2^2 a_1^3 a_3^3 \\ a_3^1 & a_3^2 & a_1^3 a_2^3 a_3^1 a_3^2 \end{vmatrix} = 0, \tag{25}$$

or

$$\begin{vmatrix} a_1^1 & a_1^1 a_1^3 a_2^2 a_3^2 & a_1^3 \\ a_2^1 & a_1^2 a_2^1 a_2^3 a_3^2 & a_2^3 \\ a_3^1 & a_1^2 a_2^2 a_2^2 a_3^1 & a_3^3 \end{vmatrix} = 0, \tag{26}$$

or

$$\begin{vmatrix} a_1^1 a_1^3 a_2^2 a_3^2 & a_1^2 & a_1^3 \\ a_1^2 a_2^1 a_2^3 a_3^2 & a_2^2 & a_2^3 \\ a_1^2 a_2^2 a_2^2 a_3^1 & a_3^2 & a_3^3 \end{vmatrix} = 0. \tag{27}$$

We came to the contradiction, since equation (25) (or (26), or (27)) shows that the matrix A is not the most generic. In fact, the matrices A satisfying equation (25) (or (26), or (27)) form an 8-parameter subfamily in the 9-parameter family of the most generic matrices A. $\square$

Remark. As we noted earlier, if an almost Grassmannizable web $AGW(6,3,2)$ is of maximum 2-rank (i.e., it admits one abelian 2-equation), then it is necessary that condition (7) holds. Since our web admits an abelian 2-equation (13), condition (7) should be satisfied identically. A simple calculation shows that in fact this is true: by (24), condition (7) is reduced to the same equation (25) (or (26), or (27)) which holds for an almost Grassmannizable web $AGW(6,3,2)$.

6. We present now another (analytic) proof of Proposition 1.

First, we remind that a web $W(d,n,r)$ is said to be *parallelizable* if it is equivalent to a web $W(d,n,r)$ formed by d foliations of parallel $(n-1)r$-dimensional planes of (nr)-dimensional affine space A^{nr}.

The following general criteria of parallelizability is valid: *A web $W(d,n,r)$ is parallelizable if and only all its basis affinors $\underset{a\sigma}{\lambda}{}^i_j$, $a=n+2,\dots,d$; $\sigma=1,\dots,n-1$, are covariantly constant on X^{nr} in an affine connection γ and its $(n+1)$-subweb $[1,2,\dots,n]$ defined by the foliations X_α, $\alpha=1,2,\dots,n$, is parallelizable.*

The proof of this theorem is similar to that of Theorem 7.2.2 in [4] (p. 311) proved there for webs $W(4,2,r)$.

Proof. In fact, taking exterior derivatives of equations (14), we find that

$$d\underset{\alpha}{\omega}{}^i = 0, \quad \alpha = 1,2,3;\ i = 1,2. \tag{28}$$

Comparing equations (28) with the structure equations (6) of a general web $W(6,3,2)$, we see that

$$\omega_j^i = 0, \quad \underset{\alpha\beta}{a}{}^i_{jk} = 0.$$

Since the basis affinors (23) of a web defined by equations (8) and (10) are constant and $\omega_j^i = 0$, it follows that

$$\nabla \underset{a\sigma}{\lambda}{}_j^i = d \underset{a\sigma}{\lambda}{}_j^i - \underset{a\sigma}{\lambda}{}_k^i \omega_j^k + \underset{a\sigma}{\lambda}{}_j^k \omega_k^i = 0,$$

i.e., the basis affinors $\underset{a\sigma}{\lambda}{}_j^i$ are covariantly constant on $\mathbb{R}^6$. By Theorem 1, this implies that the webs $W(6, 3, 2)$ of maximum 2-rank defined by equations (8) and (10) are parallelizable. □

The following corollary immediately follows from Proposition 1 and Theorem 2:

Corollary 3. *Equations* (8) *and* (10) *define a* 9*-parameter family of parallelizable, not almost Grassmannizable webs of maximum* 2*-rank.*

7. As we indicated earlier, there is a 9-parameter family of parallelizable, not almost Grassmannizable webs of maximum 2-rank. Such webs can be obtained by choosing a nonsingular matrix A with the only condition that the determinant on the left-hand side of equation (25) (or (26) or (27)) does not vanish.

For any $a = 1, 2, 3, 4, 5, 6, 7, 8$, it is easy to find a-parameter subfamilies of this 9-parameter family of webs.

For example, an 8-parameter family B_8 of such webs is defined by the matrices A with $a_1^3 = 0$; a 7-parameter family B_7 of such webs is defined by the matrices A with $a_1^2 = a_1^3 = 0$; and a 6-parameter family B_6 of such webs is defined by the matrices A with $a_1^3 = a_2^1 = a_3^2 = 0$. In each of these cases, it is easy to check that the determinant on the left-hand side of (27) does not vanish.

For each of the subfamilies B_8, B_7, and B_6, we can define a-parameter subfamilies for any $a = 1, \ldots, 7$, by relating some of the remaining nonvanishing entries of the matrix A.

We conclude by indicating three concrete examples of webs belonging respectively to the subfamilies B_8, B_7, and B_6.

Example 1. Consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

It is easy to see that this matrix is nonsingular, and the determinant on the left-hand side of (27) is equal to 2.

A simple computation by means of equations (8) and (10) gives the following closed form equations of this web:

$$\left\{ \begin{array}{ll} x^4 = x^1 + x^2 + x^3, & y_4 = -y_1 - y_2 + y_3, \\ x^5 = x^1 + x^2 + 2x^3, & y_5 = y_2 - y_3, \\ x^6 = x^2 + x^3, & y_6 = y_1 - y_2. \end{array} \right. \tag{29}$$

Example 2. Consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that this matrix is nonsingular, and the determinant on the left-hand side of (27) is equal to 1.

A simple computation by means of equations (8) and (10) gives the following closed form equations of this web:

$$
\left\{\begin{array}{ll}
x^4 = x^1 \qquad\quad +x^3, & y_4 = \qquad\qquad y_2 - y_3, \\
x^5 = x^1 + x^2, & y_5 = -\, y_1 - y_2 + y_3, \\
x^6 = \qquad x^2 + x^3, & y_6 = -\, y^1 \qquad\quad + y_3.
\end{array}\right. \tag{30}
$$

Example 3. Consider

$$
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
$$

It is easy to see that this matrix is nonsingular, and the determinant on the left-hand side of (27) is equal to -1.

A simple computation by means of equations (8) and (10) gives the following closed form equations of this web:

$$
\left\{\begin{array}{ll}
x^4 = x^1 \quad + \quad x^3, & y_4 = \frac{1}{2}(-\, y_1 + y_2 + y_3), \\
x^5 = x^1 + x^2, & y_5 = \frac{1}{2}(-\, y_1 - y_2 + y_3), \\
x^6 = \qquad x^2 + x^3, & y_6 = \frac{1}{2}(\quad y^1 - y_2 - y_3).
\end{array}\right. \tag{31}
$$

8. Finally note that if in equations (8) and (10) the indices α and β take values $1, 2, \ldots, n$, and A is an arbitrary $n \times n$ matrix, then these equations produce a $(2n)$-web $W(2n, n, 2)$ which is *a parallelizable, not almost Grassmannizable* $(2n)$*-web of maximum* 2*-rank one.*

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